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Optimal State Feedback Control Design of Half-Car Active Suspension System

Nur Uddin, *Member, IAENG,* Auralius Manurung, and Rahmat Nur Adi Wijaya

***Abstract*—An optimal control design for active suspension system of ground vehicle is presented. Objective of the active suspension system is to improve the vehicle performance in particularly on the ride comfort. The optimal control design is done by applying linear quadratic regulator (LQR), where the vehicle suspension is approached by a half-car model. The LQR formulates the control design problem into a optimizing problem for minimizing a quadratic cost function. Solving the optimizing problem results in an optimal states feedback control that is being applied in the active suspension system. Performance of the active suspension system is demonstrated through numerical simulations together with a passive suspension system. Evaluation of the simulation results shows an advantage of active suspension system by improving the ride comfort up to 94.81% of the passive suspension system.**

***Index Terms*—Active suspension, system modeling, control design, optimal control.**

I. INTRODUCTION

A suspension system applied in vehicle to overcome a degradation of vehicle performance due to road disturbances. The road disturbances are resulted by an interaction of the vehicle moving wheels and the road roughness. These disturbances results in vehicle body motions, such as heaving, pitching, and rolling. These motions may decrease the vehicle performance, e.g., ride comfort, ride safety, and handling. Therefore, the suspension system is applied to isolate the vehicle body from the motions due to road disturbances. There are different kind of suspension systems that can be classified into three types: passive, active, and semi-active [1]–[3].

The passive suspension system has two main components: spring and damper [4]–[6]. The use of spring and damper converts the road disturbances into damped oscillations on the vehicle body. The passive suspension system works well in stabilizing the vehicle vibration and has been applied in commercial vehicle for many years. Performance of the passive suspension system is determined by the values of spring constant and damping constant. Both constant values are calculated based on a value of the vehicle mass. However, the vehicle mass is varying in practice, for example due to variation of the vehicle loads, including passenger and cargo. This becomes a difficultly to maintain performance of the suspension system.

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Control system communities introduce a concept of vibration control by using a state feedback control system that is known as the active vibration control system [7]–[9]. Applying the concept on suspension system results in an active suspension system. An active element is utilized in the active suspension system to generate force for stabilizing the vibration. This active active element is also known as the actuator. A servo-hydraulic is an example of actuator applied in the active suspension system [10].

Studies on the active suspension system has been presented since 1960s [5]–[7]. The studies results show significant improvements on ride comfort, handling, and stability of the vehicle compared to the passive suspension system. The active suspension systems in those study were developed through: 1) system modelling, 2) control design, and 3) performance evaluation. The system modeling is done to obtain dynamics of the vehicle suspension. There are three common models applied in the vehicle suspension studies: quarter-car, half-car, and full-car models. The quarter-car model is used to represent one degree of freedom (DOF) suspension dynamics, while the half-car and full-car models are applied to represent two and three DOF suspension dynamics. Selection of the applied model depends on the study scope and interest, for examples: the quarter-car model in [11]–[14], the half-car model in [15], and the full-car model in [16].

Optimal control is one of the most popular control design method in active suspension system studies [11], [13], [17]. Other control methods are also applicable in active suspension system design, for examples: fuzzy control [18], proportional integral and derivative (PID) control [19], model predictive control (MPC) [20], [21], and adaptive backstepping control [22]. An advanced optimal control method in active suspension system has also been presented by including preview information [16], [23]–[25]. Those presented studies shows the superiority of active suspension system in improving vehicle performance compared to the passive suspension system. However, the active suspension system is not widely applied in commercial vehicles. Most of the commercial vehicles still uses passive suspension system. Feasibility and practical implementation of the active suspension system are still an open research problem.

A comprehensive study on an optimal control design for vehicle active suspension system is presented in this study. It is presented a detail derivation of suspension system dynamics that results into a state space equation. The vehicle suspension system is approached by a half-car model and the Newton's second law is applied to derive dynamic equations of the model. An optimal states feedback control is designed using the LQR method and applied in the active suspension system. Performance of the active suspension system is evaluated through a comparison to a passive suspension system.

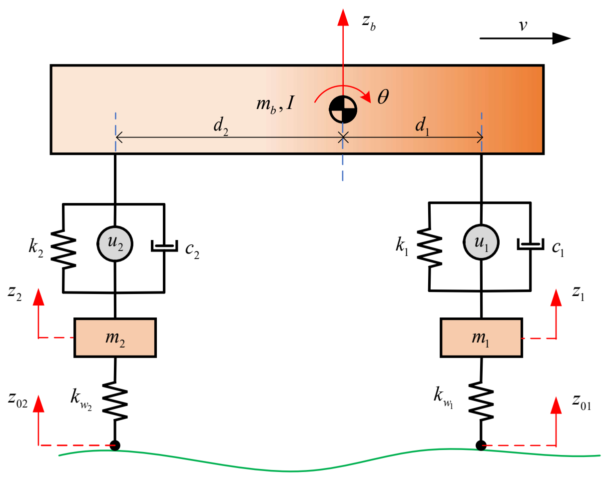

Fig. 1. The half-car suspension model.

Performances of both suspension system are numerically demonstrated through numerical simulated in a computer. Presentation of the paper is organized as follows. Section I describes an introduction and motivation of the research work. Section II describes the modeling of the suspension system. Section III discusses the optimal control design for the active suspension system. Section IV presents the simulation scenarios and simulation results in evaluating the suspension performance. Finally, Section V provides conclusion of this study .

II. Vehicle Suspension System Dynamics

A half car model of vehicle suspension system is presented in Figure 1. Mass of the vehicle is grouped into three masses: the vehicle body mass m_b, the front wheel mass m_1, and the rear wheel mass m_2. The vehicle body is supported by two suspensions connected to the wheels. Each suspensions is represented by a spring with stiffness k, an active element for generating force u, and a damper with damping coefficient c. Therefore both suspensions are active suspension. While the active element is not available, the suspension is a passive suspension. Mass of the suspension is relatively small compared to the vehicle body mass and the wheel mass and therefore is ignored. Tire of the wheel is simply modelled by air spring with stiffness coefficient k_w. The vehicle velocity is indicated by a vector v. The subscript 1 in the model notation indicates the vehicle front part, while the subscript 2 represents the rear part.

The Newton's second law is applied to derive the suspension system dynamics based on the free body diagram shown in Figure 2. Applying the law on the vehicle body mass results in the following equations:

$$m_b \ddot{z}_b = f_1 + f_2 \quad (1)$$
$$I\ddot{\theta} = -d_1 f_1 + d_2 f_2, \quad (2)$$

where z_b is the vertical displacement of vehicle body mass, I is the vehicle body inertia, θ is the pitching angle, f_1 is the vertical force at the front point, f_2 is the vertical force at the rear vehicle point, d_1 is the distance of the front wheel to the vehicle's center of mass, and d_2 is the distance of the rear wheel to the vehicle's center of mass. Both vertical forces are defined as follows:

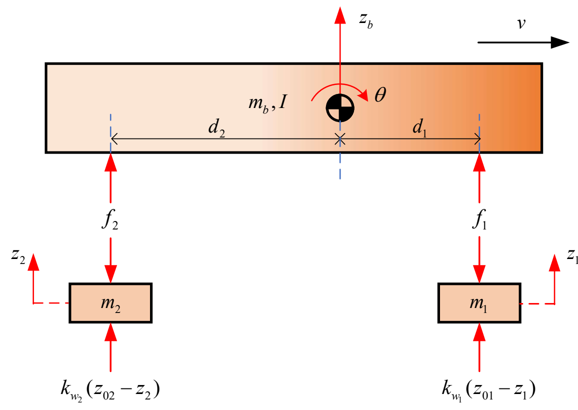

Fig. 2. Free body diagram of the half-car suspension model.

$$f_1 = u_1 - k_1(z_b - z_1 - d_1\theta) - c_1(\dot{z}_b - \dot{z}_1 - d_1\dot{\theta}) \quad (3)$$

$$f_2 = u_2 - k_2(z_b - z_2 + d_2\theta) - c_2(\dot{z}_b - \dot{z}_2 + d_2\dot{\theta}). \quad (4)$$

The z_1 and z_2 are the vertical displacement of the front and rear wheels, respectively. Dynamics of both wheels are given as follows:

$$m_1\ddot{z}_1 = -f_1 + k_{w_1}(z_{01} - z_1) \quad (5)$$
$$m_2\ddot{z}_2 = -f_2 + k_{w_2}(z_{02} - z_2) \quad (6)$$

where z_{01} and z_{02} are the road disturbances at the front and rear wheels, respectively.

Dynamics of the half car model are expressed by the equations (1), (2), (5), and (6). Define state variables of the suspension system as follows:

$$\begin{aligned} x_1 &= z_1, & x_2 &= \dot{z}_1, \\ x_3 &= z_2, & x_4 &= \dot{z}_2, \\ x_5 &= z_b, & x_6 &= \dot{z}_b, \\ x_7 &= \theta, & x_8 &= \dot{\theta}. \end{aligned} \quad (7)$$

Substituting those state variables (7) into (3) and (4) results in:

$$f_1 = u_1 - k_1(x_5 - x_1 - d_1x_7) - c_1(x_6 - x_2 - d_1x_8) \quad (8)$$

$$f_2 = u_2 - k_2(x_5 - x_3 - d_2x_7) - c_2(x_6 - x_4 + d_2x_8). \quad (9)$$

Differentiating the state variables (7) with respect to time results in the following equations:

$$\dot{x}_1 = x_2 \quad (10)$$

$$\dot{x}_2 = -\frac{1}{m_1}u_1 + \frac{k_1}{m_1}(x_5 - x_1 - d_1 x_7) + \frac{c_1}{m_1}(x_6 - x_2 - d_1 x_8) + \frac{k_{w_1}}{m_1}(-x_1 + z_{01}) \quad (11)$$

$$\dot{x}_3 = x_4 \quad (12)$$

$$\dot{x}_4 = -\frac{1}{m_2}u_2 + \frac{k_2}{m_2}(x_5 - x_3 + d_2 x_7) + \frac{c_2}{m_2}(x_6 - x_4 + d_2 x_8) + \frac{k_{w_2}}{m_2}(-x_3 + z_{02}) \quad (13)$$

$$\dot{x}_5 = x_6 \quad (14)$$

$$\dot{x}_6 = \frac{1}{m_b}(u_1 + u_2) - \frac{k_1}{m_b}(x_5 - x_1 - d_1 x_7) - \frac{c_1}{m_b}(x_6 - x_2 - d_1 x_8) - \frac{k_2}{m_b}(x_5 - x_3 + d_2 x_7) - \frac{c_2}{m_b}(x_6 - x_4 + d_2 x_8) \quad (15)$$

$$\dot{x}_7 = x_8 \quad (16)$$

$$\dot{x}_8 = -\frac{d_1}{I}u_1 + \frac{k_1 d_1}{I}(x_5 - x_1 - d_1 x_7) + \frac{d_2}{I}u_2 + \frac{d_1 c_1}{I}(x_6 - x_2 - d_1 x_8) - \frac{d_2 k_2}{I}(x_5 - x_3 + d_2 x_7) - \frac{d_2 c_2}{I}(x_6 - x_4 + d_2 x_8). \quad (17)$$

The equations (10) to (17) can be compactly expressed in a state space form as follows:

$$\dot{x} = Ax + Bu + Dw. \quad (18)$$

where x is the system states vector, A is the system matrix, B is the input matrix, u is the input vector, D is the disturbance matrix, and w is the disturbance vector. The system matrix A is given as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & \frac{-c_1}{m_1} & 0 & 0 & \frac{k_1}{m_1} & \frac{c_1}{m_1} & \frac{-d_1 k_1}{m_1} & \frac{-d_1 c_1}{m_1} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{43} & \frac{-c_2}{m_2} & \frac{k_2}{m_2} & \frac{c_2}{m_2} & \frac{d_2 k_2}{m_2} & \frac{d_2 c_2}{m_2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_1}{m_b} & \frac{c_1}{m_b} & \frac{k_2}{m_b} & \frac{c_2}{m_b} & a_{65} & a_{66} & a_{67} & a_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} \end{bmatrix}$$

where a_{ij} in the matrix denotes the matrix element at row i and column j that are defined as follows:

$$\begin{array}{ll} a_{21} = \frac{-(k_1+k_{w_1})}{m_1}, & a_{43} = \frac{-(k_2+k_{w_2})}{m_2}, \\ a_{65} = \frac{-(k_1+k_2)}{m_b}, & a_{66} = \frac{-(c_1+c_2)}{m_b}, \\ a_{67} = \frac{d_1 k_1 - d_2 k_2}{m_b}, & a_{68} = \frac{d_1 c_1 - d_2 c_2}{m_b}, \\ a_{81} = -\frac{d_1 k_1}{I}, & a_{82} = -\frac{d_1 c_1}{I}, \\ a_{83} = \frac{d_2 k_2}{I} & a_{84} = \frac{d_2 c_2}{I}, \\ a_{85} = \frac{(d_1 k_1 - d_2 k_2)}{I}, & a_{86} = \frac{d_1 c_1 - d_2 c_2}{I} \\ a_{87} = \frac{-d_1^2 k_1 - d_2^2 k_2}{I}, & a_{88} = \frac{-d_1^2 c_1 - d_2^2 c_2}{I}. \end{array}$$

Those vectors and matrices are defined as follows:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ \frac{-1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{-1}{m_2} \\ 0 & 0 \\ \frac{1}{m_b} & \frac{1}{m_b} \\ 0 & 0 \\ \frac{-d_1}{I} & \frac{d_2}{I} \end{bmatrix}, u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ \frac{k_{w_1}}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{k_{w_2}}{m_2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{and } w = \begin{bmatrix} z_{01} \\ z_{02} \end{bmatrix}.$$

III. Optimal Control Design

A states feedback control is applied in the active suspension system to stabilize vibrations on the vehicle body due to road disturbance. Optimal control is applied to design the state feedback control in this study and presented as follows.

Define the state feedback control law for the vehicle suspension system (18) as follows:

$$u = -Kx, \quad (19)$$

where u is the input vector that expresses the control command, K is the control gain matrix and x is the system states. Substituting (19) into (18) results in:

$$\dot{x} = (A - BK)x + Dw. \quad (20)$$

By defining a new matrix:

$$A_c = A - BK, \quad (21)$$

the (20) can be expressed by:

$$\dot{x} = A_c x + Dw. \quad (22)$$

The (22) is the closed loop system of (18), where A_c is the closed loop system matrix. Stability of the closed loop system is determined by the eigenvalues of A_c. The closed loop system is asymptotically stable if all eigenvalues of A_c have negative real part. Such kind of the matrix is known as a Hurwitz matrix.

The (21) shows that the matrix A_c are dependent to the matrices A, B, and K. Since the matrices A and B are representing the vehicle parameters values, tuning the both matrices requires a physical adjustment on the vehicle components, which is not practical. On the other hand, the matrix K is adjustable by tuning the control parameters value. The matrix K is therefore designed to make the matrix A_c to be Hurwitz. The control gain matrix K is obtained through a control design process. It can be done by using one of the available control design methods and this study applies the linear quadratic regulator (LQR). The LQR is an optimal control method that calculates the control gain matrix K by minimizing a quadratic cost function [26]–[28].

Main objective of the active suspension system in this study is to improve ride comfort of the vehicle. The active suspension system is desired to minimize the heaving and pitching motions of the vehicle. The heaving motion is represented by the vertical position z_b, while the pitching motion is represented by the pitching angle θ. It is realized that the suspension moving space is limited. Therefore, it is also desired to minimize the suspension displacement. The suspension displacement is related to the vehicle ride safety such reducing the displacement implicates an improvement on the vehicle ride safety [29]. For accommodating the main objective and the requirement, define an output vector y to represents displacement of both suspensions, the vehicle heaving motion, and the vehicle pitching motion. The front suspension displacement is defined by:

$$y_1 = z_b - z_1 - d_1\theta \tag{23}$$

while the rear suspension displacement is expressed by:

$$y_2 = z_b - z_2 + d_1\theta. \tag{24}$$

Therefore, the output vector y can be defined as follows:

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} z_b - z_1 - d_1 \\ z_b - z_2 + d_1 \\ z_b \\ \theta \end{bmatrix}. \tag{25}$$

Stating y as a function of the system state variables, x, results in the following equation:

$$y = \begin{bmatrix} x_5 - x_1 - d_1x_7 \\ x_5 - x_3 + d_2x_7 \\ x_5 \\ x_7 \end{bmatrix}. \tag{26}$$

The output vector y in (26) is a linear combination of the system states vector x. Therefore, it can be expressed as follows:

$$y = Cx, \tag{27}$$

where C is known as the system output matrix and defined as follows:

$$C = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & -d_1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{28}$$

Minimizing the output vector y needs to be done by using a minimum effort. The effort is the forces generated by the active elements that are represented by the input vector u. The minimization is done based on the following cost function:

$$J = \frac{1}{2}\int_0^{\infty} \left(y^T\bar{Q}y + u^TRu\right) dt, \tag{29}$$

where $\bar{Q}$ and R are the weighting matrices. The matrix $\bar{Q}$ is a symmetric positive semi definite matrix, while the matrix R is a positive definite matrix. Substituting (27) into (29) results in:

$$J = \frac{1}{2}\int_0^{\infty} \left(x^TC^T\bar{Q}Cx + u^TRu\right) dt. \tag{30}$$

For simplifying the expression, define a new matrix

$$Q = C^T\bar{Q}C \tag{31}$$

and substituting it into (30) such that results in:

$$J = \frac{1}{2}\int_0^{\infty} \left(x^TQx + u^TRu\right) dt. \tag{32}$$

The optimal control problem is defined as a problem of finding the control input u that minimizes the cost function J. Mathematics derivation to solve to the optimal control problem can be found in many optimal control literature. The following derivation of the optimal control solution refers to [26] and is explained as follows.

The cost function J in (32) is minimized through minimizing the following Hamiltonian function:

$$H = \frac{1}{2}\left(x^TQx + u^TRu\right) + \lambda^T(Ax + Bu) \tag{33}$$

where H is the Hamiltionian function and λ is the costate. The Hamiltonian function is minimized by the following two conditions:

$$\frac{\partial H}{\partial x} = -\dot{\lambda} \tag{34}$$

$$\frac{\partial H}{\partial u} = 0. \tag{35}$$

The first condition is achieved by:

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -Qx - A^T\lambda \tag{36}$$

and the second condition is achieved by:

$$u = -R^{-1}B^T\lambda. \tag{37}$$

Substituting (37) into (18) and ignoring the disturbance results in:

$$\dot{x} = Ax - BR^{-1}B^T\lambda. \tag{38}$$

Define the costate λ as follows:

$$\lambda = Px, \tag{39}$$

where P is a symmetric matrix. Substituting the costate into (36) results in:

$$\dot{P}x + P\dot{x} = -Qx - A^TPx, \tag{40}$$

while substituting the costate into (38) results in:

$$\dot{x} = Ax - BR^{-1}B^TPx. \tag{41}$$

Substituting (41) into (40) yields in:

$$(\dot{P} + PA + A^TP - PBR^{-1}B^TP + Q)x = 0. \tag{42}$$

Non-trivial solution of (42) is obtained by solving the time-differential equation:

$$\dot{P} = -(PA + A^TP - PBR^{-1}B^TP + Q). \tag{43}$$

which is known as the Riccati equation. Steady state of the Riccati equation is given by:

$$0 = PA + A^TP - PBR^{-1}B^TP + Q \tag{44}$$

that is known as the algebraic Riccati equation (ARE). The matrix P is obtained by solving the ARE. While the matrix P is found, solution of the optimal control problem is obtained by substituting P into (39) and then substituting (39) into (37) such that results in:

$$u = -Kx, \tag{45}$$

where K is the control gain matrix given by:

$$K = R^{-1}B^TP. \tag{46}$$

The (45) is the optimal control solution for minimizing the cost function (29). The (46) and (44) show that the control gain matrix K is a function of the weighting matrices Q and R. Therefore, the control gain matrix K can be tuned by adjusting the elements of matrices Q and R.

IV. SIMULATION

The optimal states feedback control derived in the previous section is applied in an active suspension system of ground vehicle. Dynamics of the vehicle are approached by the half-car model given in (18). The active suspension system has a main objective on improving the vehicle ride comfort, while the vehicle ride safety is maintained or even more improved. The vehicle ride comfort and ride safety are expressed by the output vector y given in (25) that consists of four elements: y_1, y_2, y_3, and y_4. The y_1 and y_2 denotes displacements of the front and the rear suspensions, respectively, that related to the ride safety. The y_3 and y_4 denotes the vehicle heaving motion and the the vehicle pitching motion that correspondences to the ride comfort.

Cost of each output variable of the system is defined as follows:

$$J_{y_i} = \frac{1}{2}\int_0^\infty \sigma_i y_i^2 dt \tag{47}$$

and the system output cost is defined by:

$$J_y = \sum_{i=1}^{4} J_{y_i} = \sum_{i=1}^{4}\left(\frac{1}{2}\int_0^\infty \sigma_i y_i^2 dt\right) \tag{48}$$

where y_i is the i^{th} output variable, σ_i is a positive constant representing weighting factor of the output variable y_i, J_{y_i} is the cost function of the output variable y_i, and J_y is the system output cost.

The active suspension system has two inputs, u_1 and u_2. The u_1 is the force generated by actuator of the front active-suspension, while the u_2 is the force generated by actuator of the rear active-suspension. Cost of the system input is defined by the following equations:

$$J_u = \sum_{k=1}^{2}\left(\frac{1}{2}\int_0^\infty \rho_k u_k^2 dt\right). \tag{49}$$

where u_k is the k^{th} input variable, ρ_i is a positive constant representing the weighting factor of input variable u_i, and J_u is the cost of system input.

Total cost of the suspension system is defined as follows:

$$J = J_y + J_u \tag{50}$$

where J is the total performance index of the suspension system. Substituting (48) and (49) into (50) results in:

$$J = \sum_{i=1}^{4}\left(\frac{1}{2}\int_0^\infty \sigma_i y_i^2 dt\right) + \sum_{k=1}^{2}\left(\frac{1}{2}\int_0^\infty \rho_k u_k^2 dt\right) \tag{51}$$

that can be expressed into the following equation:

$$J = \frac{1}{2}\int_0^\infty \left(y^T W_1 y + u^T W_2 u\right) dt, \tag{52}$$

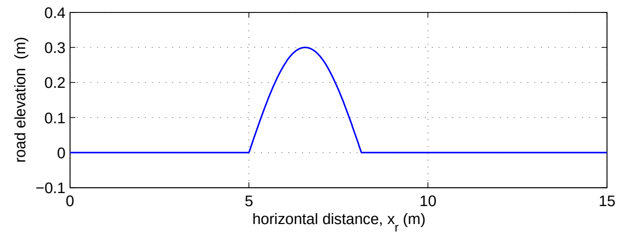

Fig. 3. Road profile and the road disturbance on both vehicle wheels.

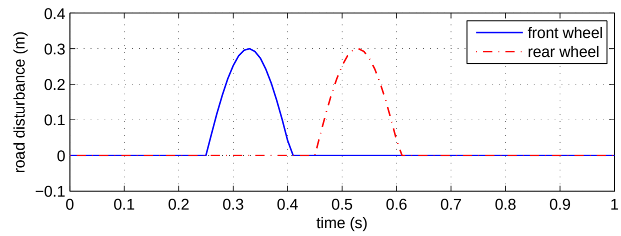

Fig. 4. Road profile and the road disturbance on both vehicle wheels.

where W_1 is a diagonal matrix with the matrix element $W_1(i,i) = \sigma_i$, W_2 is a diagonal matrix with the matrix element $W_2(k,k) = \sigma_k$, y is the output vector, and u is the input vector. The (52) and (29) are similar and both are equal if $\bar{Q} = W_1$ and $R = W_2$. Therefore, the matrices $\bar{Q}$ and R are designed to be diagonal matrices in this study. The matrix $\bar{Q}$ is defined as follows:

$$\bar{Q} = \begin{bmatrix} \bar{q}_1 & 0 & 0 & 0 \\ 0 & \bar{q}_2 & 0 & 0 \\ 0 & 0 & \bar{q}_3 & 0 \\ 0 & 0 & 0 & \bar{q}_4 \end{bmatrix}, \tag{53}$$

where $\bar{q}_i$ are the weighting factor of the system output y_i defined in (25) for $i = 1, 2, 3, 4$. While for the matrix R, it is defined as follows:

$$R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}, \tag{54}$$

where r_1 is the weighting factor for system input u_1 and r_2 is the weighting factor for system input u_2.

A computer program is built to demonstrate performance of the vehicle suspension systems. The computer program simulates the vehicle move at a constant speed 20 m/s and pass a bump with amplitude of 30 cm. For the simulation, the bump is approached by the following function:

$$z_r = \begin{cases} 0, & \text{for } 0 < x_r < 5 \\ a_r \sin(x_r - 5), & \text{for } 5 \le x_r \le 5 + \pi \\ 0, & \text{for } x_r > 5 + \pi \end{cases} \tag{55}$$

where a_r is the bump amplitude, x_r is the horizontal road position, and z_r is the road elevation. The bump road profile is shown in Figure 3. While passing the bump, the vehicle is excited by a road disturbance through the front wheels and followed by the rear wheels. The road disturbances on both wheels are shown in the Figure 4. Parameters of both passive and active suspension systems, and the vehicle for the simulation are listed in Table I.

TABLE I
VEHICLE PARAMETERS

Parameter	Symbol	Value	Unit
Vehicle			
front wheel mass	m_1	40	kg
rear wheel mass	m_2	40	kg
body mass	m_b	1400	kg
body inertia	I_b	2000	Nm
front tyre stiffness	k_{w_1}	2×10^5	N/s
rear tyre stiffness	k_{w_2}	2×10^5	N/s
Active suspension			
spring stiffness	k_1, k_2	2×10^4	N/s
damping coefficient	c_1, c_2	2600	Ns/m
Passive Suspension			
spring stiffness	k_1, k_2	2×10^4	N/s
damping coefficient	c_1, c_2	2600	Ns/m

TABLE II
WEIGHTING MATRICES OF ACTIVE SUSPENSION SYSTEM

Simulation Name	Weighting Matrices
Sim 1	$\bar{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$
Sim 2	$\bar{Q} = \begin{bmatrix} 10^3 & 0 & 0 & 0 \\ 0 & 10^3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$
Sim 3	$\bar{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10^3 & 0 \\ 0 & 0 & 0 & 10^3 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$
Sim 4	$\bar{Q} = \begin{bmatrix} 10^3 & 0 & 0 & 0 \\ 0 & 10^3 & 0 & 0 \\ 0 & 0 & 10^3 & 0 \\ 0 & 0 & 0 & 10^3 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$
Sim 5	$\bar{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 10^{-9} & 0 \\ 0 & 10^{-9} \end{bmatrix}.$
Sim 6	$\bar{Q} = \begin{bmatrix} 10^3 & 0 & 0 & 0 \\ 0 & 10^3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 10^{-9} & 0 \\ 0 & 10^{-9} \end{bmatrix}.$
Sim 7	$\bar{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10^3 & 0 \\ 0 & 0 & 0 & 10^3 \end{bmatrix}, R = \begin{bmatrix} 10^{-9} & 0 \\ 0 & 10^{-9} \end{bmatrix}.$
Sim 8	$\bar{Q} = \begin{bmatrix} 10^3 & 0 & 0 & 0 \\ 0 & 10^3 & 0 & 0 \\ 0 & 0 & 10^3 & 0 \\ 0 & 0 & 0 & 10^3 \end{bmatrix}, R = \begin{bmatrix} 10^{-9} & 0 \\ 0 & 10^{-9} \end{bmatrix}.$

Since the states feedback control of active suspension system is designed using the optimal control, the active suspension performance is determined by the weighting matrices of the cost function (29). Varying the weighting matrices will result in different performance. It is demonstrated in this study by presenting eight sets of different weighting matrices as listed in Table II. Each weighting matrices sets is used to calculate a control gain matrix of the optimal state feedback controller. The resulted controller is applied in the active suspension system and simulated together with the passive suspension system. Therefore, eight simulations are performed and sequentially named as the Sim 1 to Sim 8. Performance of the suspension systems are calculated based on the system cost (32), where u is a zero vector for the passive suspension. The better performance is indicated by the lower cost.

The eight simulations are carried out and the resulted costs are presented in the Table III and Table IV. The results of each simulations are discussed as follows:

a) The Sim 1 is done by selecting both weighting matrices $\bar{Q}$ and R equal to the identity matrices. The simulation result shows that costs of both active and passive suspension systems are equal. This indicates that the actuator of active suspension system did not generate a significant control force. The active suspension system is dominated by the works of spring and damper. This is confirmed by a small value of the system input cost J_u. Adjustment of the weighting matrices is required to improve the performance of active suspension system.
b) Increasing weighting matrix for system output $\bar{Q}$ of Sim 1 is done for the matrix elements $\bar{q}_1$ and $\bar{q}_2$ in the Sim 2. Both are increased 10^3 times of the values in the Sim 1. The $\bar{q}_1$ and $\bar{q}_2$ are the weighting factors for the front and rear suspension displacements, respectively. Increasing values of both weighting factors indicates a more emphasizing for reducing the suspension displacements. The simulation results in the same cost of both active and passive suspension systems. Cost of the system input is still very small and incomparable to the system output cost.
c) Increasing the weighting matrix $\bar{Q}$ of Sim 1 by more emphasizing on the heaving and pitching motions is done in the Sim 3. The matrix elements $\bar{q}_3$ and $\bar{q}_4$ are increased 10^3 times. However, the simulation results of Sim 3 shows that this increment does not show any different on the system-output cost between the active and passive suspension systems. The system-input cost of active suspension system is still very small and insignificantly influences to the total cost.
d) The Sim 4 increases the weighting matrix Q by multiplying the elements of $\bar{Q}$ of the Sim 1 by 10^3. The simulation results of Sim 4 show the same values of system-output cost of both active and passive suspension system. Cost of the system input is still very small and imbalance to be compared to the system output cost.
e) The simulation results of the active suspension system in the Sim 1 to Sim 4 show that the values of J_u and J_y are very imbalance, where the ratio of J_u and J_y in the order of 10^{-9}. This is a hint for tuning the matrix R. Control force generated by the actuator is determined by a control law given in (45). The control law shows that the generated control force is proportional to the control gain K and the system states x. Since all of the system states are counted to determine cost of the system output, the simulation results of the Sim 1 to Sim 4 indicate that the very small control force u was due to the small control gain K. Therefore, the control gain has to be increased. According to (46), increasing the control gain can be done by reducing the matrix R. In this Sim 5, the weighting input matrix R is adjusted by decreasing the diagonal elements of R with scaling factor 10^{-9} of the R in Sim 1. The simulation results of Sim 5 shows that the active suspension system

TABLE III
SIMULATION RESULTS: COST OF ACTIVE AND PASSIVE SUSPENSION SYSTEMS

Simulation Name	Suspension Type	Cost						
		J_{y1}	J_{y2}	J_{y3}	J_{y4}	J_y	J_u	J
Sim 1	active	0.71	0.62	0.41	0.10	1.85	1.53×10^{-9}	1.85
Sim 1	passive	0.71	0.62	0.41	0.10	1.85	0	1.85
Sim 2	active	714.65	622.24	0.41	0.10	1337.40	6.19×10^{-4}	1337.40
Sim 2	passive	714.65	622.24	0.41	0.10	1337.40	0	1337.40
Sim 3	active	0.71	0.62	412.97	98.53	512.85	2.19×10^{-4}	512.85
Sim 3	passive	0.71	0.62	412.98	98.53	512.85	0	512.85
Sim 4	active	714.65	622.24	412.98	98.53	1848.39	1.56×10^{-3}	1848.39
Sim 4	passive	714.65	622.24	412.98	98.53	1848.40	0	1848.40
Sim 5	active	0.52	0.42	0.17	0.09	1.19	0.11	1.31
Sim 5	passive	0.71	0.62	0.41	0.10	1.85	0	1.85
Sim 6	active	57.32	23.76	0.46	0.19	81.73	43.50	125.23
Sim 6	passive	714.65	622.24	0.41	0.10	1337.40	0	1337.40
Sim 7	active	0.73	0.53	9.43	17.13	27.83	2.70	30.52
Sim 7	passive	0.72	0.62	412.98	98.53	512.85	0	512.85
Sim 8	active	210.98	70.27	199.45	136.15	616.85	19.25	636.10
Sim 8	passive	714.65	622.24	412.98	98.53	1848.40	0	1848.40

TABLE IV
COST RATIO AND PERFORMANCE IMPROVEMENT OF THE ACTIVE SUSPENSION SYSTEM TO THE PASSIVE SUSPENSION SYSTEM

Simulation Name	Cost ratio (%)			Improvement (%)		
	Ride Safety	Ride Comfort	Total Cost	Ride Safety	Ride Comfort	Total Performance
Sim 1	100	100	100	0	0	0
Sim 2	100	100	100	0	0	0
Sim 3	100	100	100	0	0	0
Sim 4	100	100	100	0	0	0
Sim 5	70.68	50.98	70.81	29.32	49.02	29.19
Sim 6	6.06	127.45	9.36	93.94	-27.45	90.64
Sim 7	94.04	5.19	5.95	5.97	94.81	94.05
Sim 8	21.04	65.61	34.41	78.96	34.39	65.59

results in less system-output cost and less total cost. Costs of the system-input and the system-output are close to balance with the ratio of J_u and J_y about 0.1. The active suspension system of Sim 5 improves the ride safety 29.32%, the ride comfort 49.02%, and the total performance 29.19% compared to the passive suspension system. Although the active suspension results in better performance, more improvement of the total performance is still desired.

f) A re-adjustment of the weighting matrices of Sim 5 is presented in Sim 6. The Sim 6 adjusts the matrix $\bar{Q}$ while the matrix R remains to be the same as in the Sim 5. Sim 6 emphasizes on the front and rear suspension deflection by increasing the weighting factor $\bar{q}_1$ and $\bar{q}_2$ of Sim 5 to be 10^3 times. The active suspension system results in much lower costs on both suspension deflections but slightly higher costs on the heaving and pitching motions. The Sim 6 achieves improvements of 93.94% on the ride safety, -27.45% on the ride comfort, and 90.64%. The active suspension system of Sim 6 makes a very good improvement on the ride safety but decreases the ride comfort.

g) Re-tuning on the weighting matrices of Sim 5 is also presented Sim 7 by emphasizing on the vehicle ride comfort performance. The Sim 7 modifies the values of $\bar{q}_3$ and $\bar{q}_4$ of matrix $\bar{Q}$ while the other weighting matrices elements are the same as in the Sim 5. The simulation results of Sim 7 show improvements on the ride safety 5.97%, ride comfort 94.81%, and total performance 94.05% by using the active suspension system.

h) Another adjustment on weighting matrices of Sim 5 is presented in Sim 8. The diagonal element of matrix $\bar{Q}$ are increased 10^3 time while the matrix R is fixed as in the Sim 5. According to the simulation results of Sim 8, the active suspension system makes improvement on the ride safety 78.96%, ride comfort 34.39%, and total performance 65.59%.

According to the simulation results, the best performance of ride comfort was achieved by the active suspension system of Sim 7, while the active suspension system with least suspension deflection was resulted in the Sim 6. Time responses of the suspension systems in Sim 6 and Sim 7 are shown in Figure 5 to Figure 7. The Figure 5 shows deflections of the front and rear suspension. The active suspension system of Sim 6 produced in the least deflections for both front and

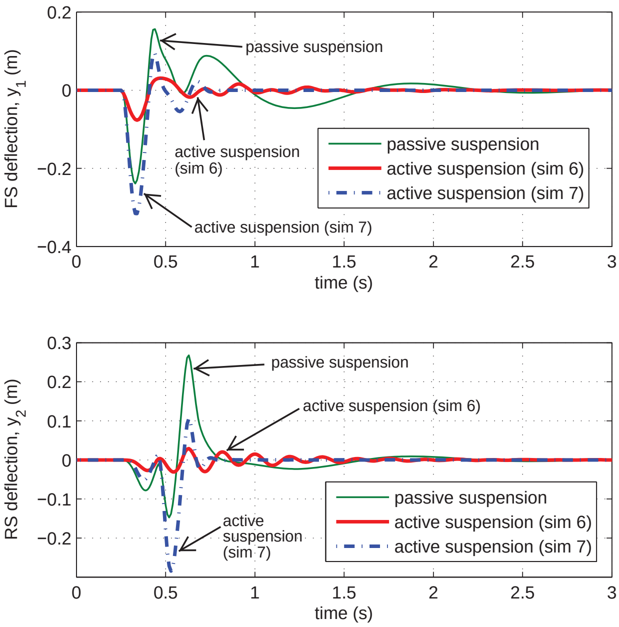

Fig. 5. The front suspension (FS) deflection and the rear suspension (RS) deflection of passive and active suspension systems.

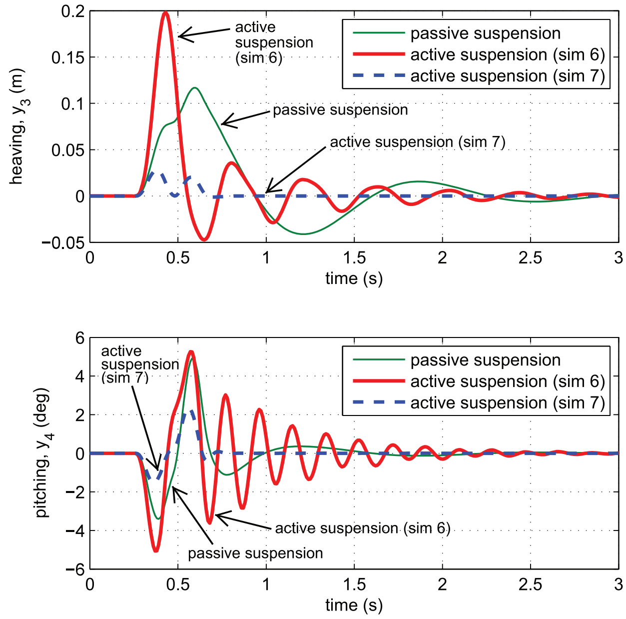

Fig. 6. The vehicle body motions of using passive and active suspension systems.

rear suspensions among the three suspension systems. The Figure 6 shows the vehicle heaving and pitching motions. The active suspension system of Sim 7 results in the least heaving and pitching motions among the three suspension system. The least heaving and pitching motions implicates the best ride comfort. The active suspension system of Sim 6 exhibits oscillations of heaving and pitching motions. This oscillations reduce the ride comfort of the vehicle. A comparison of the required control force for both active suspension systems are presented in the Figure 7. The figure shows that the active suspension system of Sim 6 requires more control force than the active suspension system of Sim 7. Considering the implementation cost, this makes implementation of the active suspension system of Sim 6 be more expensive.

V. CONCLUSIONS

An optimal state feedback control design for active suspension system has been presented. The control design was done based on an half-car suspension model and applying the LQR control design method. Performance evaluation of the suspension system was carried out through computer simulations. Performance of the active suspension system is determined by the weighting matrices of the LQR cost function. The cost function includes the weighting matrix of system output and the weighting matrix of the system input. Eight variations of the weighting matrices were presented and simulated. The results show that: 1) weighting matrices of the system output and the system input have to be initially tuned such that costs of the system output and the system input are balance, 2) giving more weighting on the ride comfort resulted in the better performance than giving more weighting on the suspension displacement. The best active suspension system for ride comfort was achieved in the Sim 7 by improving the vehicle ride comfort 94.81%, the vehicle

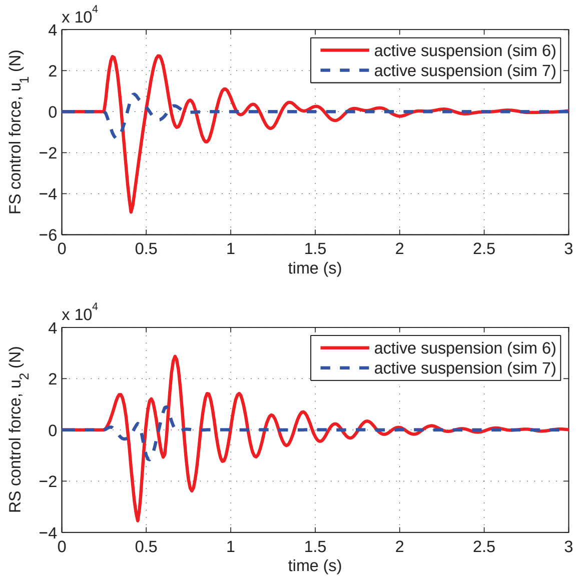

Fig. 7. Control forces generated by the front and rear actuators.

ride safety 5.97%, and the total suspension-performance 94.81%.

This study was done by modeling the tyre as spring. A more realistic and reliable tyre model should be considered for a further study, for an example by applying a tyre model presented in [30].

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